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Correlation functions in disordered systems

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Recently, we found that the correlation between the eigenvalues of random Hermitian matrices exhibits universal behavior. Here we study this universal behavior and develop a diagrammatic approach which enables us to extend our previous work to the case in which the random matrix evolves in time or varies as some external parameters vary. We compute the current-current correlation function, discuss various generalizations, and compare our work with the work of other authors. We study the distribution of eigenvalues of Hamiltonians consisting of a sum of a deterministic term and a random term. The correlation between the eigenvalues when the deterministic term is varied is calculated.

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I. INTRODUCTION

We have been studying correlations between energy eigenvalues in random matrix theory [1–4] in an attempt to uncover possible universal behavior in disordered systems. Let us begin by summarizing the main results of our previous work [5,6]. Consider an ensemble of $N\times N$ Hermitian matrices φ defined by the probability distribution

$$P(\varphi)=\frac{1}{Z}e^{-N^2\mathcal{H}(\varphi)}\ , \tag{1.1}$$

with

$$\mathcal{H}(\varphi)=\frac{1}{N}\operatorname{tr}V(\varphi) \tag{1.2}$$

for any even polynomial V and with Z fixed by the normalization $\int d\varphi\, P(\varphi)=1$. Define the Green's functions

$$G(z)\equiv\left\langle\frac{1}{N}\operatorname{tr}\frac{1}{z-\varphi}\right\rangle\ , \tag{1.3}$$

$$G(z,w)\equiv\left\langle\frac{1}{N}\operatorname{tr}\frac{1}{z-\varphi}\frac{1}{N}\operatorname{tr}\frac{1}{w-\varphi}\right\rangle\ , \tag{1.4}$$

and so forth, where

$$\langle 0(\varphi)\rangle\equiv\int d\varphi\, 0(\varphi)P(\varphi)\ . \tag{1.5}$$

The density of eigenvalues is then given by

$$\rho(\mu)=\left\langle\frac{1}{N}\operatorname{tr}\delta(\mu-\varphi)\right\rangle=\frac{-1}{\pi}\operatorname{Im}G(\mu+i\epsilon) \tag{1.6}$$

and the correlation between eigenvalues, by

$$\rho(\mu,\nu)=\left\langle\frac{1}{N}\operatorname{tr}\delta(\mu-\varphi)\frac{1}{N}\operatorname{tr}\delta(\nu-\varphi)\right\rangle$$
$$=\left[-\frac{1}{4\pi^2}\right][G(++)+G(--)-G(+-)-G(-+)]\ , \tag{1.7}$$

with the obvious notation

$$G(\pm,\pm)\equiv G(\mu\pm i\epsilon,\nu\pm i\delta) \tag{1.8}$$

(signs uncorrelated).

In the large N limit, $\rho(\mu,\nu)\to\rho(\mu)\rho(\nu)$, and thus it is customary to define the connected correlation

$$\rho_c(\mu,\nu)\equiv\rho(\mu,\nu)-\rho(\mu)\rho(\nu)\ . \tag{1.9}$$

Note that the factors of N are chosen in our definitions such that the interval $[-a,+a]$ over which $\rho(\mu)$ is nonzero is finite (i.e., of order N^0) in the large N limit.

For applications to disordered systems, φ is often thought of as the Hamiltonian. Its eigenvalues then describe the energy levels of the system. In some applications, φ is related to the transmission matrix [7]. The density of and correlation between its eigenvalues tell us about the conductance fluctuation in disordered metals.

The density of eigenvalues has long been known in the literature [8] to have the form

$$\rho(\mu)=\frac{1}{\pi}P(\mu)\sqrt{a^2-\mu^2}\ , \tag{1.10}$$

where $P(\mu)$ is a polynomial of degree $2p-2$ if the potential V is of degree $2p$. The polynomial $P(\mu)$ and the end point of the spectrum a depend on V in a complicated way. In our recent work, we have focused on the correlation between eigenvalues.

We have obtained the following results [5,6], all in the large N limit.

(1) Using the method of orthogonal polynomials, we determine

$$\rho_c(\mu,\nu)=-\left[\frac{a}{4N}\right]^2(\mu-\nu)^{-2}[f(\mu)f(\nu)]^{-1}$$
$$\times[\{\cos[\varphi(\mu)]-\cos[\varphi(\nu)]\}(\cos\{N[h(\mu)+h(\nu)]\}+\cos\{N[h(\mu)-h(\nu)]\})$$
$$+\{\sin[\varphi(\mu)]-\sin[\varphi(\nu)]\}\sin\{N[h(\mu)+h(\nu)]\}+\{\sin[\varphi(\mu)]+\sin[\varphi(\nu)]\}\sin\{N[h(\mu)-h(\nu)]\}]^2. \quad (1.11)$$

The various functions $f(\mu)$, $\varphi(\mu)$, and $h(\mu)$ are given in [5] and depend on V. Thus, this rather complicated expression depends on V in detail.

(2) With $\mu-\nu$ close together, of order N^{-1} times some number large compared to unity so that there can still be a finite number of eigenvalues between μ and ν, and with both μ and ν at a finite distance from the end points of the spectrum, we obtain the universal result

$$K(\mu,\nu)\to\frac{\sin[2\pi N\delta\mu\rho(\bar{\mu})]}{2\pi N\delta\mu}\ , \quad (1.12)$$

where $\bar{\mu}\equiv\frac{1}{2}(\mu-\nu)$ and $\delta\mu\equiv\frac{1}{2}(\mu-\nu)$. This means that the correlation function takes the form

$$\rho(\mu,\nu)=\rho(\mu)\rho(\nu)\left[1-\left[\frac{\sin x}{x}\right]^2\right]\ , \quad (1.13)$$

$$x=2\pi N\delta\mu\rho(\bar{\mu})\ . \quad (1.14)$$

This result has long been known in the literature. For $(\mu-\nu)$ small, this implies that $\rho(\mu,\nu)$ vanishes as

$$\rho(\mu,\nu)\sim\tfrac{1}{3}\pi^2\rho(\bar{\mu})^4[N(\mu-\nu)]^2\ , \quad (1.15)$$

as expected from the Van der Monde determinant in the measure. Note in this context that it is incorrect to replace $\sin^2 x$ by its average $\frac{1}{2}$, as is sometimes done in the literature.

(3) The wild oscillation of $\rho_c(\mu,\nu)$ is entirely as expected since between μ and ν finitely separated there are in general $O(N)$ eigenvalues. Thus, it is natural to smooth $\rho_c(\mu,\nu)$ by integrating over intervals $\delta\mu$ and $\delta\nu$ large compared to $O(N^{-1})$ but small compared to $O(N^0)$ centered around μ and ν, respectively. We then obtain [5]

$$\rho_c^{\text{smooth}}(\mu,\nu)=\frac{-1}{2N^2\pi^2}\frac{1}{(\mu-\nu)^2}\frac{(a^2-\mu\nu)}{[(a^2-\mu^2)(a^2-\nu^2)]^{1/2}}\ . \quad (1.16)$$

We find this result rather remarkable since the density $\rho(\mu)$ is completely nonuniversal. The only dependence on V appears through a.

(4) We have also computed the three- and four-point connected correlation functions. When the oscillations in these functions are smoothed over, we found that they vanish identically to $O(N^{-3})$ and $O(N^{-4})$, respectively. We conjectured that the smoothed p-point connected correlation function similarly vanishes to $O(N^{-p})$.

(5) We can show that the results in paragraphs (1), (2), and (3) hold for an ensemble much more general than the one defined in (1.1) and (1.2), namely, an ensemble defined with

$$H(\varphi)=\frac{1}{N}\mathrm{Tr}V(\varphi)+\frac{1}{N^2}\mathrm{Tr}W_1(\varphi)\mathrm{Tr}W_2(\varphi)$$
$$+\frac{1}{N^3}\mathrm{Tr}X_1(\varphi)\mathrm{Tr}X_2(\varphi)\mathrm{Tr}X_3(\varphi)+\cdots\ , \quad (1.17)$$

with W, X, Y,. . . arbitrary polynomials. Indeed, it is easy to see [6] that this ensemble can be further generalized by replacing, for example, the third term in (1.13) by

$$\frac{1}{N^3}\sum_\alpha \mathrm{Tr}X_1^\alpha(\varphi)\mathrm{Tr}X_2^\alpha(\varphi)\mathrm{Tr}X_3^\alpha(\varphi)\ , \quad (1.18)$$

with X_i^α a polynomial. This ensemble appears to us to be the most general ensemble invariant under unitary transformations, i.e.,

$$P(U^\dagger\varphi U)=P(\varphi)\ , \quad (1.19)$$

except for some rather singular examples.

(6) Wigner [1] also studied nonunitary ensembles, for example, an ensemble of matrices φ whose matrix elements take on the value $\pm v/\sqrt{N}$ with equal probability. Using a renormalization group [9] inspired approach, we can show that $\rho(\mu)$ for this class of matrix is universal and equal to the $\rho(\mu)$ for the simple Gaussian ensemble defined with $V(\varphi)=(m^2/2)\varphi^2$ in (1.2), namely,

$$\rho(\mu)=\frac{1}{\pi}\sqrt{a^2-\mu^2}\ , \quad (1.20)$$

that is, Wigner's well-known semicircle law.

(7) In [6] we outline an argument showing that the results stated in paragraphs (1), (2), (3), and (4) also hold for the nonunitary ensemble mentioned in paragraph (6).

Some of our results appear to overlap with results obtained in the recent literature [10,11]. In particular, our smoothed universal connected correlation (1.16) appears to have been discovered also by Beenakker in an interesting work [11], although we have not yet seen a full derivation.

Given our correlation functions (1.11), (1.13), and (1.16), we can proceed to determine the mean square fluctuation of physical quantities such as conductance. For any quantity $A(\varphi)$ defined by the trace of some function of φ, the mean square fluctuation is clearly given by

$$\mathrm{var}A\equiv\langle A^2\rangle-\langle A\rangle^2=\int d\mu\, d\nu\, \rho_c(\mu,\nu)A(\mu)A(\nu)\ . \quad (1.21)$$

We would like to emphasize that it is incorrect to put for $\rho_c(\mu,\nu)$ the smoothed correlation

$$\rho_c^{\text{smooth}}(\mu,\nu)\propto\frac{1}{(\mu-\nu)^2}$$

given in (1.16). The singularity in $\rho_c^{\text{smooth}}(\mu,\nu)$ as

$\mu-\nu\to 0$ would produce a divergent integral. Within our discussion, there is of course no difficulty whatsoever, since $\rho_c^{\rm smooth}(\mu,\nu)$ was derived with the explicit proviso that $\mu-\nu$ is of $O(N^0)$ and the true $\rho_c(\mu,\nu)$ given in (1.11) is perfectly smooth as $\mu-\nu\to 0$. Calculation of the variance of the conductance given in the recent literature [11] appears to us to involve simply replacing ρ_c by $\rho_c^{\rm smooth}$ without justification.

From the definitions for $\rho(\mu)$ and $\rho_c(\mu,\nu)$, it is easy to derive the response of $\rho(\mu)$ under a change in the potential $V(\nu)$, as pointed out by Beenakker [11]:

$$\delta\rho(\mu)=-N^2\int d\nu\,\rho_c(\mu,\nu)\delta V(\nu)\ . \tag{1.22}$$

Again, it would be tempting to replace [11] $\rho_c(\mu,\nu)$ by $\rho_c^{\rm smooth}(\mu,\nu)$. However, this illegitimate procedure would lead to a divergent integral. The variation $\delta\rho(\mu)/\delta V(\nu)$ appears to pose a rather tedious calculation with P and a in (1.10) depending on V in a complicated way.

We took V to be an even polynomial for simplicity, so that the density of eigenvalues is a symmetric function between the end points $-a$ and $+a$. It is a simple matter to shift the spectrum. Clearly, if we replace ϕ in V by $\phi-dI$ (with I the unit matrix and d the shift), the density of eigenvalues would be nonzero between $c=-a+d$ and $b=a+d$. The universal correlation function is then trivially shifted to read

$$\rho_c^{\rm smooth}(\mu\nu)=-\frac{1}{2N^2\pi^2}\frac{1}{(\mu-\nu)^2}\frac{[-bc+\frac{1}{2}(b+c)(\mu+\nu)-\mu\nu]}{[(b-\mu)(\mu-c)(b-\nu)(\nu-c)]^{1/2}}\ . \tag{1.23}$$

In Sec. II we will develop a diagrammatic method which will enable us to study "time"-dependent correlation between the eigenvalues and which, when the time dependence is suppressed, allows us to recover many of the results mentioned above. In Sec. III we compute the current-current correlation function. In Sec. IV we study a class of Hamiltonians consisting of the sum of a deterministic term and a random term. The correlation between the eigenvalues when the deterministic term is varied is calculated.

II. "TIME"-DEPENDENT CORRELATION

In this paper we introduce a diagrammatic method to study the correlation between eigenvalues for a time-dependent ensemble of Hermitian matrices with a probability distribution defined by

$$P(\varphi)=\frac{1}{Z}\exp-\int_{-T}^{T}dt\,{\rm Tr}\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2+V(\varphi)\right]\ . \tag{2.1}$$

We take $T\to\infty$. Here the matrix $\varphi(t)$ depends on time, or, more generally, on some external parameter we are allowed to vary. Physically, we may apply our results obtained below to disordered systems in which the disorder may vary. In going from (1.1) to (2.1) we are moving from zero-dimensional field theory (i.e., an integral) to one-dimensional field theory (i.e., Euclidean quantum mechanics). In the language of string theory, we move from a central charge $c=0$ theory to a $c=1$ theory. As a byproduct, we show how some of our previous results mentioned in Sec. I may be recovered as a special case. For ease of presentation, we will take $V(\varphi)=(m^2/2)\varphi^2$ to be Gaussian and indicate below how our results may be generalized.

The one-point Green's function, or, more generally the propagator

$$G_{ij}(z)\equiv\left\langle\left(\frac{1}{z-\varphi(t)}\right)_{ij}\right\rangle \tag{2.2}$$

can be readily determined since due to time translation invariance, it does not in fact depend on time. Using the usual Feynman diagram expansion, we find immediately that in the large N limit $G_{ij}(z)$ is given by planar Feynman diagrams (generalized rainbow) as indicated in Fig. 1(a).

It is perhaps useful, borrowing the terminology of large N quantum chromodynamics (QCD) from the particle physics literature, to speak of the single line in Fig. 1(a)

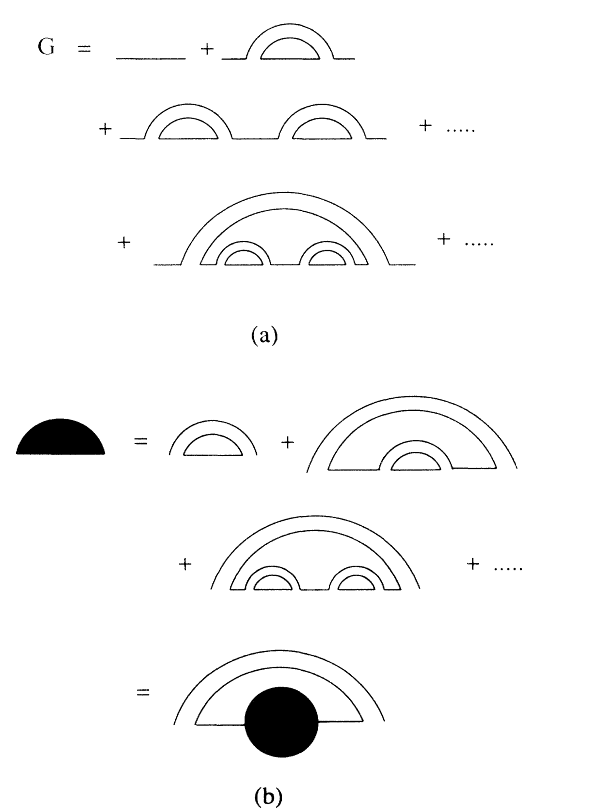


FIG. 1. (a) Feynman diagram expansion for the Green's function $G(z)$. (b) The generalized rainbow equation for the one-particle-irreducible self-energy $\Sigma(z)$.

as representing quark propagators and the double lines as gluon propagators. The quark propagator is given simply by $1/z$ while the gluon propagator is given by

$$D_{ij,kl}(t)\equiv\langle\varphi_{ij}(t)\varphi_{kl}(0)\rangle$$
$$=\delta_{il}\delta_{jk}\frac{1}{N}\int\frac{d\omega}{2\pi}\frac{e^{i\omega t}}{\omega^2+m^2}=\delta_{il}\delta_{jk}\frac{1}{2Nm}e^{-m|t|}\ . \tag{2.3}$$

We will now immediately generalize to arbitrary time dependence by replacing (2.3) by

$$D_{ij,kl}(t)=\delta_{il}\delta_{jk}\frac{\sigma^2}{N}e^{-u(t)}\ . \tag{2.4}$$

For instance, the time dependence in (2.1) may be changed in such a way that the "momentum space" propagator ω^2+m^2 in (2.4) is replaced by $1/(\omega^2+\gamma|\omega|+m^2)$ or $1/(\omega^4+\alpha\omega^2+m^2)$, for example. In other words, in (2.1) we can replace $\frac{1}{2}(d\varphi/dt)^2$ by $\varphi K(d/dt)\varphi$ with K any reasonable function. Our only requirement is that u is a smooth function of t and does not blow up as t goes to zero.

Introducing as usual the one-particle irreducible self-energy $\Sigma_{ij}(z)$, we can write the generalized rainbow integral equation [Fig. 1(b)]

$$\Sigma(z)=\sigma^2\frac{1}{z-\Sigma(z)}=\sigma^2G(z)\ . \tag{2.5}$$

Here we have used the fact, which is immediately obvious from examining the Feynman diagrams, that $\Sigma_{ij}(z)$ is equal to $\delta_{ij}\Sigma(z)$ and the fact that the gluon propagator only appears at equal time

$$D_{ij,kl}(0)\equiv\delta_{il}\delta_{jk}\frac{1}{N}\sigma^2\ .$$

Note that the quark does not know about time. Solving the quadratic equation for Σ, we obtain the Green's function as defined in (1.3):

$$G(z)=\frac{1}{2\sigma^2}(z-\sqrt{z^2-4\sigma^2})\ . \tag{2.6}$$

Taking the absorptive part, we recover immediately Wigner's semicircle law as given in (1.20).

Incidentally, within this diagrammatic approach, band matrices can be treated immediately. Let the matrices φ be restricted so that φ_{ij} vanishes unless $|i-j|<bN/2$ with $b<1$. Such matrices describe, for example, the hopping of a single electron on a one-dimensional lattice with random hopping amplitudes. The essential feature is that from each site the electron can hop to $O(1/N)$ sites. Looking at the Feynman diagrams, we see that in the generalized rainbow integral equation we simply restrict the range of summation from N to bN and thus, instead of (2.3), we obtain

$$\Sigma(z)=b\sigma^2\frac{1}{z-\Sigma(z)}=b\sigma^2G(z)\ . \tag{2.7}$$

Thus, we have the same distribution of eigenvalues with a suitable redefinition of the end points.

Let us now move on to the connected two-point Green's function

$$G_c(z,w,t)\equiv\left\langle\frac{1}{N^2}\,\mathrm{tr}\frac{1}{z-\varphi(t)}\,\mathrm{tr}\frac{1}{w-\varphi(0)}\right\rangle_C$$
$$=\frac{1}{N^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{z^{m+1}w^{n+1}}\langle\mathrm{tr}\varphi^m(t)\,\mathrm{tr}\varphi^n(0)\rangle_C\ . \tag{2.8}$$

Henceforth, for the sake of notational clarity we will set σ to unity; it can always be recovered by dimensional analysis. Diagrammatically, the expression for $G_c(z,\omega,t)$ can be described as two separate quark loops, carrying "momentum" z and w, respectively, interacting by emitting and absorbing gluons [see Fig. 2(a)].

With a Gaussian distribution for φ, we can readily "Wick-contract" the expression $\langle\mathrm{tr}\varphi^m(t)\mathrm{tr}\varphi^n(0)\rangle_C$. Let us begin by ignoring contractions within the same trace (in which case m and n are required to be equal). In the large N limit, the dominant graphs [see Fig. 2(b)] are given essentially by "ladder graphs" (with one crossing) which immediately sum to

$$N^2G_c(z,w)=\frac{1}{(zw)^2}\frac{1}{\left[1-\frac{1}{zw}\right]^2}\ . \tag{2.9}$$

We next include Wick contractions within the same trace in $\langle\mathrm{tr}\varphi^m\mathrm{tr}\varphi^n\rangle$. We see that graphically these contractions describe vertex and self-energy corrections. The vertex corrections can be immediately summed: The expression in (2.9) is to be multiplied by two factors, the factor

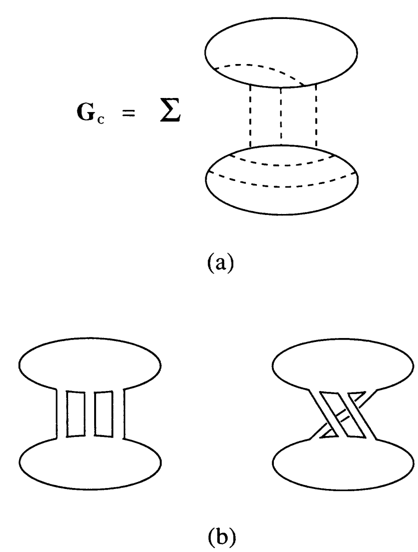


FIG. 2. (a) Feynman diagram expansion for the connected two-point Green's function. The dotted lines here represent the "gluon" propagator. (b) Some typical low order Feynman diagrams contributing to the connected two-point Green's function.

$$\frac{1}{\left[1-\frac{1}{z^2}\right]^2} \tag{2.10}$$

and a similar factor with z replaced by w. Finally, according to our earlier discussion, self-energy corrections are included immediately by replacing the bare quark propagator $1/z$ by the dressed propagator $G(z)$ (and similarly for $1/w$, of course.) We obtain finally the remarkably compact result

$$N^2G_c(z,w) = \frac{1}{[1-G(z)G(w)]^2}\left[\frac{G^2(z)}{1-G^2(z)}\right]\left[\frac{G^2(w)}{1-G^2(w)}\right] . \tag{2.11}$$

Finally, we have to put in the time dependence, which we have ignored so far. We note first of all that the vertex and self-energy corrections contain no time dependence, since there the gluons always begin and end on the same quark line. Thus, the time-dependent-two-point connected Green's function can be written down immediately as

$$N^2G_c(z,w,t)=\frac{e^{-u(t)}}{[1-e^{-u(t)}G(z)G(w)]^2} \times\left[\frac{G^2(z)}{1-G^2(z)}\right]\left[\frac{G^2(w)}{1-G^2(w)}\right] . \tag{2.12}$$

To obtain the connected correlation function between energy eigenvalues, we have to take the double absorptive part of $G_c(z,w,t)$ as indicated in (1.7). It is most convenient to introduce angular variables: From (2.6) we see that we may write

$$G(\mu+i\epsilon)=-i\eta e^{i\eta\theta} , \tag{2.13}$$

where η is equal to the sign of ϵ and

$$\sin\theta\equiv\mu/a . \tag{2.14}$$

Similarly, we write

$$G(\nu+i\delta)=-i\xi e^{i\xi\phi} , \tag{2.15}$$

with ξ equal to the sign of δ and

$$\sin\phi\equiv\nu/a . \tag{2.16}$$

As μ and ν vary over their allowed ranges, from $-a$ to $+a$, θ and ϕ range vary from $-\pi/2$ to $\pi/2$.

We can now readily compute [in the notation of (1.8)]

$$8N^2G_c(++)=\frac{1}{\cos\theta\cos\phi}\left\{\frac{1}{[1+\cosh u\cos(\theta+\phi)-i\sinh u\sin(\theta+\phi)]}\right\} . \tag{2.17}$$

Proceeding in this way, we obtain one of our main results:

$$-16\pi^2N^2\rho_c(\mu,\nu)=\frac{1}{\cos\theta\cos\phi}\left\{\frac{1+\cosh u\cos(\theta+\phi)}{[\cosh u+\cos(\theta+\phi)]^2}+(\phi\rightarrow-\phi+\pi)\right\} . \tag{2.18}$$

Note that crossing symmetry ($\mu\leftrightarrow\nu$, $t\rightarrow-t$) clearly holds. For $t=0$, that is, $u=0$ (since any nonzero $u(0)$ can be absorbed into σ^2), we recover immediately our previous result (1.16), obtained by the orthogonal polynomial method. Note that time acts as a regulator for the singularity when $\mu=\nu$: For time not equal to zero, we can set $\mu=\nu$ without difficulty and obtain

$$8\pi^2N^2\rho_c(\mu,\mu)=\frac{1}{\cos^2\theta}\frac{1}{u^2}+\cdots . \tag{2.19}$$

For $\mu\neq\nu$ and small time $u\ll\theta-\phi$, we have

$$-16\pi^2N^2\rho_c(\mu,\nu)=\frac{1}{\cos\theta\cos\phi}\left\{\frac{1}{1+\cos(\theta+\phi)}\left[1-\frac{1-\frac{1}{2}\cos(\theta+\phi)}{1+\cos(\theta+\phi)}u^2\right]+(\phi\rightarrow-\phi+\pi)\right\} . \tag{2.20}$$

In the long time limit, $u\rightarrow\infty$, we find

$$4\pi^2N^2\rho_c(\mu,\nu)\rightarrow e^{-u}\tan\theta\tan\phi . \tag{2.21}$$

As expected, the correlation vanishes exponentially in time. Notice, however, that a memory of the spatial correlation is retained even at arbitrarily large time. We thus conclude that the density-density correlation is universal in space for all time, in the sense that it does not depend on V at all.

We note that in the diagrammatic approach used here, the correlation function is "automatically" smoothed, namely, that we obtain directly the smoothed correlation in (1.16) rather than the detailed correlation in (1.11) that we obtained using the orthogonal polynomial method. An interchange of limits is responsible. Here, in computing $G(z,w)$, we first take N to infinity and then let z and w approach the real axis. When N is set to infinity, the poles of $G(z,w)$ on the real axis corresponding to the individual eigenvalues merge into a cut, and thus the "short distance" or the detailed structure of the eigenvalue spectrum is smoothed over. In contrast, in our previous work, we use orthogonal polynomials to calculate $\rho_c(\mu,\nu)$ directly without going into the complex plane. In effect, we let z and w sit on the real axis before taking N to infinity and thus the discrete pole structure of the eigenvalue spectrum is visible.

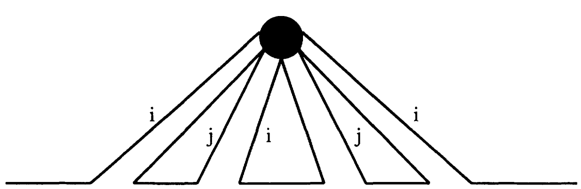
In this context we may also mention that in the recent literature on the subject, particularly in the work of Altshculer and collaborators that we will cite below, the focus is on the short distance behavior of the correlation function, that is to say, the behavior of the correlation function in the regime specified for (1.12) and (1.13). The diagrammatic approach allows us to study the long distance regime specified for (1.16). The orthogonal polynomial method, however, tells us about the correlation over all distance scales. In this sense, the orthogonal polynomial method is more powerful and informative.

Having analyzed the Gaussian case, we now discuss how our results could hold more generally. We distinguish between the "trace class" defined in (1.1) and the "Wigner class" defined in (6).

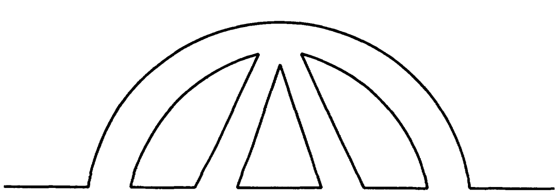
For the Wigner class, let us focus on the example in which the probability of the distribution matrix element φ_{ij} is given by

$$P(\varphi_{ij}) \propto e^{-N^2[|\varphi_{ij}|^2-(v^2/N)]^2} , \tag{2.22}$$

corresponding essentially to the example mentioned in item (6) in Sec. I (see [6]). The quartic interaction $\sim N^2|\varphi_{ij}|^4$ would contribute to Feynman diagrams such as the one in Fig. 3(a). Counting powers, we see that this graph is of order $N^{-4}NN^2=N^{-1}$ and so is suppressed relative to the graphs in Fig. 1(a). Reasoning along this line, we see immediately that the distribution of eigenvalues is universal, a long-known result that we also derived recently using a renormalization group inspired approach [6]. As a bonus, we obtain immediately in the present diagrammatic approach that the correlation function is also universal. (Incidentally, this result is not at all easy to obtain with the renormalization group approach of [6].)



(a)



(b)

FIG. 3. Non-Gaussian corrections to the propagator in (a) the Wigner class and (b) the trace class.

For the trace class, the interaction, for example, the quartic term in $V(\varphi)$, would generate Feynman diagrams such as the one in Fig. 3(b). Counting powers of N, we see that this graph is in no way suppressed relative to the graphs in Fig. 1(a). This is in fact a gratifying conclusion as we know from [8] [see Eq. (1.10)] that in the trace class, in sharp contrast to the Wigner class, the distribution of eigenvalues is in fact not universal. It appears to us that within the diagrammatic approach, it would be rather involved to demonstrate the universality of the correlation function for the trace class, namely, the result we obtained in [5] using the method of orthogonal polynomials. We would have to show that the effects of the arbitrary polynomial interactions contained in V in (1.1) can be summed up and absorbed completely in the end point value a of the spectrum. We find it remarkable that in this subject results easily obtained in one approach are apparently rather difficult to prove in another.

III. CURRENT-CURRENT CORRELATIONS

As is well known, the ensemble in (1.1) may be thought of as describing the statistical mechanics of a gas in 1+0 dimensional space time. The partition function

$$Z=\int d\varphi\, e^{-N\,\mathrm{tr}V(\varphi)}$$
$$=C\int d\lambda_1 \cdots d\lambda_N \times\exp\left[-N\sum_i V(\lambda_i)+\sum_{i<j}\ln(\lambda_i-\lambda_j)^2\right] , \tag{3.1}$$

where C is an irrelevant overall constant. The logarithmic term comes from the well-known Jacobian connecting $d\varphi$ to $d\lambda_1 d\lambda_2 \ldots d\lambda_N$. Regarding λ_i as the position of the "ith particle" on the real line, we see that (3.1) describes a one-dimensional gas of N particles interacting with each other via a logarithmic repulsion while confined by an external potential $V(\lambda)$. Intuitively then, it becomes entirely clear that the density of the gas $\rho(\mu)$ has no reason to be universal: It should certainly depend on V. It is less clear why the change in density $\delta\rho(\mu)$ at μ due to a change in the potential $\delta V(\nu)$ at ν should be universal, and indeed, this universality holds only when we smooth over the discrete character of the gas.

The generalization in (2.1) then corresponds to allowing the particles to move in a 1+1-dimensional space time. With the density operator defined by

$$\rho(\mu,t)=\frac{1}{N}\sum_i \delta[\mu-\lambda_i(t)] , \tag{3.2}$$

we clearly have the conservation law

$$\frac{\partial\rho}{\partial t}+\frac{\partial J}{\partial\mu}=0 , \tag{3.3}$$

with the current operator defined by

$$J(\mu,t)=\frac{1}{N}\sum_i \frac{d\lambda_i}{dt}\delta(\mu-\lambda_i(t)) . \tag{3.4}$$

Thus, we are led to study the current-current correlation function

$$\langle J(\mu,t)J(\nu,0)\rangle = -\frac{\partial^2}{\partial t^2}\int_{-a}^{\mu} d\mu' \int_{-a}^{\nu} d\nu' \langle \rho(\mu',t)\rho(\nu',0)\rangle_C \,. \quad (3.5)$$

Note that $\langle JJ\rangle$ is connected by definition.

The double integral in (3.5) may be explicitly evaluated. We find that in effect $\rho_c(\mu,\nu)$ may be written as

$$-4\pi^2N^2\rho_c(\mu,\nu)=\frac{\partial}{\partial\mu}\frac{\partial}{\partial\nu}\ln\left[\frac{\cosh u+\cos(\theta+\phi)}{\cosh u-\cos(\theta-\phi)}\right]. \quad (3.6)$$

In particular, at equal time, we have

$$-4\pi^2N^2\rho_c(\mu,\nu) = \frac{\partial}{\partial\mu}\frac{\partial}{\partial\nu}\ln\left[\frac{a^2-\mu\nu+\sqrt{(a^2-\mu^2)(a^2-\nu^2)}}{a^2-\mu\nu-\sqrt{(a^2-\mu^2)(a^2-\nu^2)}}\right]. \quad (3.7a)$$

An equivalent form reads

$$2\pi^2N^2\rho_c(\mu,\nu) = \frac{\partial}{\partial\mu}\frac{\partial}{\partial\nu}\ln\left[\frac{\left[\frac{a-\mu}{a+\mu}\right]^{1/2}-\left[\frac{a-\nu}{a+\nu}\right]^{1/2}}{\left[\frac{a-\mu}{a+\mu}\right]^{1/2}+\left[\frac{a-\nu}{a+\nu}\right]^{1/2}}\right]. \quad (3.7b)$$

The current-current correlation function then follows immediately:

$$\begin{aligned}4\pi^2N^2\langle J(\mu,t)J(\nu,0)\rangle &= \frac{\partial^2}{\partial t^2}\ln\left[\frac{\cosh u+\cos(\theta+\phi)}{\cosh u-\cos(\theta-\phi)}\right]\\ &= \frac{\sinh u}{\cosh u+\cos(\theta+\phi)}\ddot{u} + \frac{[1+\cosh u\cos(\theta+\phi)]}{[\cosh u+\cos(\theta+\phi)]^2}\dot{u}^2 - (\phi\to-\phi+\pi)\,. \end{aligned} \quad (3.8)$$

We have thus obtained the current-current correlation function for arbitrary separation in space and time. First, the dependence on space is universal: as a function of θ and ϕ, the current-current correlation, just like the density-density correlation from which it is derived, does not depend on the potential V.

The dependence on time, in contrast, is nonuniversal: Clearly, the dependence of u on t enters. In special cases, however, the specific functional form may be seen to drop out. From (3.8), we see that at the same point in space, that is, when $\theta=\phi$, and u small, we have

$$2\pi^2N^2\langle J(\mu,t)J(\nu,0)\rangle=\frac{\partial^2}{\partial t^2}\ln u+\cdots\,. \quad (3.9)$$

Thus, if as $t\to 0$, u vanishes as $u\to\alpha t^\gamma$, then

$$2\pi^2N^2\langle JJ\rangle\to-\gamma/t^2\,. \quad (3.10)$$

We have universality in the sense that the unknown constant α has dropped out. With the further assumption that $\gamma=1$, which is reasonable but certainly not required, we obtain the universal statement

$$2\pi^2N^2\langle JJ\rangle\to-1/t^2\,. \quad (3.11)$$

We also observe the curiosity that at $\theta=\phi=\pm\pi/2$, the current-current correlation vanishes identically for all time.

For large time, we obtain

$$\pi^2N^2\langle J(\mu,t)J(\nu,0)\rangle\to\cos\theta\cos\phi(\dot{u}^2-\ddot{u})e^{-u}\,. \quad (3.12)$$

This is, of course, not independent of (2.19).

We have learned that the universality of $\langle JJ\rangle$ at small time has already been derived by Szafer and Altschuler [12] and by Beenakker [13] using apparently rather different methods and implicitly assuming that $\gamma=1$. To our knowledge, the complete form of $\langle JJ\rangle$ in (3.8) has not appeared before in the literature.

We can immediately generalize the preceding discussion to the case where many external parameters $t_1, t_2, \ldots, t_k, \ldots$ are varied. This may be described picturesquely as a many-time world in which the conservation laws

$$\frac{\partial J^k}{\partial\mu}=\frac{\partial\rho}{\partial t_k} \quad (3.13)$$

hold where the current with respect to time t_k is defined by

$$J^k(\mu)=\sum_i\frac{\partial\lambda_i}{\partial t_k}\delta[\mu-\lambda_i(t)]\,. \quad (3.14)$$

The relation (3.5) immediately generalizes to

$$\langle J^k(\mu,t)J^l(\nu,0)\rangle = -\partial^k\partial^l\int_{-a}^{\mu}d\mu'\int_{-a}^{\nu}d\nu'\langle\rho(\mu',t)\rho(\nu',0)\rangle_C\,, \quad (3.15)$$

where $\partial^k\equiv\partial/\partial t_k$. For the special case where u is a function of $t=(\sum_k t_k^2)^{1/2}$, for example, we obtain easily that

$$2\pi^2N^2\langle J^k(\mu,t)J^l(\nu,0)\rangle\to\frac{t_kt_l}{\left[\sum_j t_j^2\right]^{1/2}} \quad (3.16)$$

for small time, with an assumption similar to the one that leads to (3.11).

IV. DETERMINISTIC PLUS RANDOM

Our diagrammatic approach allows us to study immediately the eigenvalues of a Hamiltonian of the form $H=H_0+\varphi$ which consists of the sum of a deterministic piece H_0 and a random piece φ with a probability distribution such as in (1.1). Pastur [14] has found the interesting relation that

$$G(z)=G_0(z-G(z))\,, \quad (4.1)$$

where G and G_0 are the Green's functions for H and H_0, respectively. We now show that Pastur's relation follows immediately from our diagrammatic analysis.

Let us consider the Gaussian case and let H_0, be diagonal with diagonal elements ϵ_i. Looking at the relevant Feynman diagrams, we see that we simply have to replace the inverse quark propagator z by $z-\epsilon_i$ and thus obtain

$$\Sigma(z)=\frac{1}{N}\sum_k \frac{1}{z-\epsilon_k-\Sigma(z)}=G(z)\ . \tag{4.2}$$

We recognize this as (4.1). The relation (4.1), while interestingly compact, is not terribly useful in practice as in solving for $G(z)$ one would encounter a polynomial equation of degree $N+1$.

We will now demonstrate the power of the diagrammatic approach by showing how we can immediately go beyond Pastur's relation and study correlation. Consider the following class of physical problems: Suppose we change some external parameters so that the deterministic Hamiltonian H_0 is changed to H_0'. We would like to compute the correlation between the spectra of H and $H'=H_0'+\varphi$; in other words, we would like to compute

$$G_c(z,w,H_0,H_0')\equiv\left\langle \frac{1}{N}\operatorname{tr}\frac{1}{z-H_0-\varphi}\frac{1}{N}\operatorname{tr}\frac{1}{w-H_0'-\varphi}\right\rangle_c\ , \tag{4.3}$$

where the average, as before, is over the distribution of φ.

An example of this class of problems was recently studied by Simons and Altschuler [15]. They considered the problem of a single noninteracting electron moving in a ring threaded by a magnetic flux and with the electron scattering on impurities in the ring. The magnetic flux is then changed to some other value with H changed accordingly to H'. The correlation between the spectra of H and H' is apparently of great interest in the physics of mesoscopic systems.

We see immediately from Fig. 2(b) that using the diagrammatic approach we can determine $G_c(z,w,H_0,H_0')$ quite readily. Let us define

$$g_i(z)=\frac{1}{z-\epsilon_i-\Sigma(z)}\ , \tag{4.4}$$

where $\Sigma(z)$ is the solution of (4.2). We assume that H_0' is also diagonal, with diagonal elements ϵ_i'. We define the analog of $g_i(z)$ for H_0', namely,

$$h_i(w)=\frac{1}{w-\epsilon_i'-\Sigma'(w)}\ , \tag{4.5}$$

where $\Sigma'(w)$ is the solution of the analog of (4.2) for H_0', namely, (4.2) with $\epsilon_i\to\epsilon_i'$, $z\to w$, and $\Sigma(z)\to\Sigma'(w)$. Let us also introduce the shorthand notation

$$\begin{aligned} g\cdot g&\equiv\frac{1}{N}\sum_i g_i^2\ ,\\ g\cdot h&\equiv\frac{1}{N}\sum_i g_i h_i\ ,\\ g^2\cdot h&\equiv\frac{1}{N}\sum_i g_i^2 h_i\ , \end{aligned} \tag{4.6}$$

and so forth. Then,

$$N^2G_c(z,w,H_0,H_0')=\left[\frac{g^2\cdot h^2(1-g\cdot h)+(g^2\cdot h)(g\cdot h^2)}{(1-g\cdot h)^2}\right]\times\frac{1}{1-g\cdot g}\frac{1}{1-h\cdot h}\ . \tag{4.7}$$

We see that this collapses to (2.11) when $H_0=H_0'=0$.

The number of physical situations covered by the result of this section is very large. In particular, it should allow us to verify the intuitive expectations concerning the universality of the correlations: The general belief is that if the energy scale is such that the system can explore the full extent of the disorder, one should recover the correlations for the pure random matrix correlations, irrespective of the nonrandom H_0. It is far from obvious from the explicit representation (4.7). Clearly, our result (4.7) could also be used to study questions such as localizations, or the influence of white noise in various physical situations. Thus, we believe that the result in (4.7) would prove to be of importance in studying many disordered systems.

Note added in proof. Recently, C. W. J. Beenakker [Nucl. Phys. (to be published)] and B. Eynard [Nucl. Phys. (to be published)] have given alternative derivations of the universality in (1.16). The conjecture mentioned under (4) in Sec. I has been verified by Eynard.

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APPENDIX

We have focused on random Hermitian matrices. In some physical situations, the Hamiltonian matrix is in fact real symmetric. It is well known in the literature (see, for example, [7]) that, going from the case of Hermitian random matrices to the case of real symmetric matrices, we simply insert in various formulas appropriate factors of 2. This is most easily seen by looking at (3.1): for φ real symmetric matrices we would have a factor of $\frac{1}{2}$ in front of the logarithmic repulsion. This comes about because the Jacobian connecting $d\varphi$ to $d\lambda_1 d\lambda_2\ldots d\lambda_N$ for the Hermitian case is the positive square root of the corresponding Jacobian for the real symmetric case.

Let us now sketch exceedingly briefly how this factor of 2 emerges in the diagrammatic approach. Typically, we encounter $\langle \operatorname{tr}\varphi^n \operatorname{tr}\varphi^n\rangle$. To indicate how the argument goes, let us consider only Wick contractions between the two traces. After the first contraction, we have

$$\begin{aligned} n\langle \varphi_{ij}\varphi_{\alpha\beta}\rangle\langle(\varphi^{n-1})_{ji}(\varphi^{n-1})_{\beta\alpha}\rangle &= n(\delta_{i\alpha}\delta_{j\beta}+\delta_{i\beta}\delta_{j\alpha})\langle(\varphi^{n-1})_{ji}(\varphi^{n-1})_{\beta\alpha}\rangle \\ &= 2n\,\delta_{i\beta}\delta_{j\alpha}\langle\varphi^{n-1}_{ji}\varphi^{n-1}_{\beta\alpha}\rangle \ . \end{aligned} \tag{A1}$$

The last line follows from the fact that φ is symmetric. Proceeding, we find easily that the above is equal to $2nN^n$. Note that for Hermitian matrices the expression in the parentheses would reach $\delta_{i\alpha}\delta_{j\beta}$ instead, and this accounts for the factor of 2 alluded to above.

Graphically, the fact that $\langle\varphi_{ij}\varphi_{\alpha\beta}\rangle$ is equal to $\delta_{i\alpha}\delta_{j\beta}+\delta_{i\beta}\delta_{j\alpha}$ rather than $\delta_{i\alpha}\delta_{j\beta}$ means that the double lines in the gluon propagator in the Feynman diagrams can be twisted. The direct counting of graphs becomes considerably more involved. We also note that the method of orthogonal polynomials used in [5] becomes rather complicated when we deal with real symmetric matrices.

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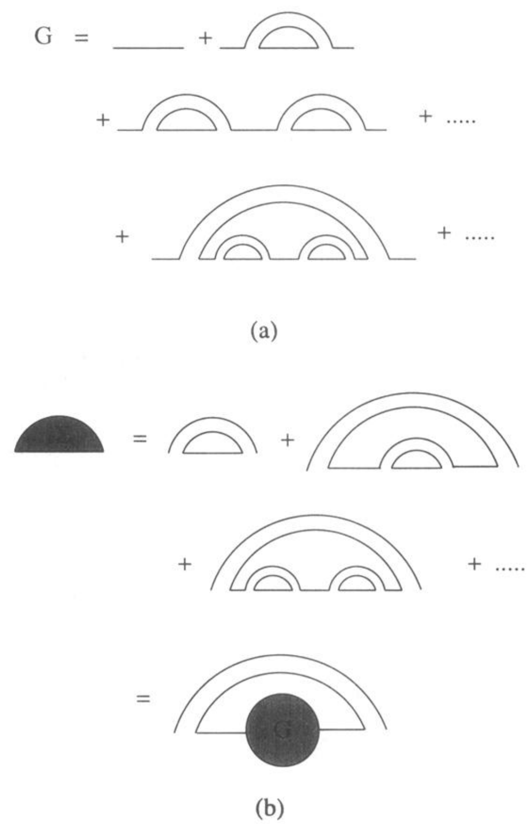

FIG. 1. (a) Feynman diagram expansion for the Green's function $G(z)$. (b) The generalized rainbow equation for the one-particle-irreducible self-energy $\Sigma(z)$.